f) **Chaotic Pendulum.**

A pendulum is formed by a point of mass m attached at the end of a rigid massless bar of length l that can rotate in a plane around the other end. When the bar form an angle θ with the vertical, under the action of the gravity, a torque $mgl\sin\theta$ is applied to the pendulum that tends to rotate it towards the vertical equilibrium position. g is the gravity constant. As the pendulum moves a friction torque $-r\theta'$ is also applied to it, that is proportional and contrary to the angular speed $\omega = \theta'$. An external oscillatory torque $h\cos\phi$ is also applied, where the angle ϕ increases linearly with time: $\phi = kt + a$. The analogue to Newton´s law for rotation establishes that:

Moment of inertia×angular acceleration = sum of all torques

$$ml^2\theta'' = -s\theta' - mglsen\theta + h\cos\phi$$

where ml^2 is the moment of inertia of the pendulum relative to its rotation point. Dividing by this quantity and calling: $q = s/ml^2$, $r = mgl/ml^2$, $b = h/ml^2$ the differential equation for the angle θ is obtained:

$$\theta'' = -q\theta' - rsen\theta + b\cos\phi$$

This second order differential equation can be transform in a system of two first order differential equations. If the angular velocity $\omega = \theta'$ is introduced, the equations for the system are:

$$\omega' = -r\omega - rsen\theta + b\cos\phi$$

$$\theta' = \omega$$

$$\phi' = k$$

It is seen that for certain values of the parameters (see Exercise 22) the state point of the system in the state space $\omega, \theta, \phi,$ is very complex. In the figure a projection in the ω, θ plane is shown. This does not correspond to the classic equilibrium schemas (attractors) mentioned in d) above (asymptotic or oscillating points, saddle points or limit cycles). They are called **strange attractors**. The observation of the trajectory does not show any periodicity or regularity and it seems totally random, in spite of the deterministic nature of the equations of the system. Experiments with the equations (see example 1 in the appendix) show that to very small differences between initial conditions corresponds quite different trajectories.

Little changes in the parameters also produce strong changes in the trajectories. This type of behavior that appears in some **non linear system** of **three or more** state variables is called **chaotic behavior** and are found in a lot of differential equation models of real and artificial systems. (See Golub 1993; Schuster).

Why many natural systems are chaotic?. The intuitive explanation is that these systems have some bifurcation states, i.e. states in which a small perturbation in one of the state variables may turn the system into different states. Such is the case of the pendulum. When by the combination of the external and internal forces it reaches very approximately an inverted vertical position with zero velocity it may be very sensitive to very small exogenous perturbations (for example the air moved by a butterfly flying some meters away) which can turn the pendulum to one side or the other which will produce very different future behaviors. So, to very similar states will succeed very different histories according to very small and imperceptible changes in the environment. If we re-start the system with the same initial condition the behavior will be different because some conditions of the environment will be different. Chaotic natural processes are not repeatable.

More subtle is the chaotic behavior of a deterministic model programmed for a computer which is a strictly deterministic machine. The random external perturbations are not in the program and the chaos might not appear. But the numerical solution of the equations are not exact and has various sources of errors. One of them are the rounding errors introduced almost in each one of the millions of numerical operations, these errors are unavoidable because of the finite representation of the real numbers. The result of that combination of errors is practically unpredictable and is similar of a noise that affects the low order digits of the values of the variables. So, to states that would be very near the bifurcation in an exact calculation, the noise may produce an effect similar to the random environmental perturbations. To very similar states near the bifurcation points will succeed different decisions to one or the other side and to this will follow very different histories of the system. But process in the computer model is fundamentally different to the natural case: if

the same run of the model is repeated, the chaotic history will be repeated in exactly the same way because the operations done are the same and so is the “noise” they produce.
On the other hand if during the computing we introduce a small instantaneous change in one of the variables (simulating the effect of a passing butterfly) the posterior history of the system will be very different because to a next bifurcation state will follow a different decision than in a run without the perturbation.
Many complex physical or social systems may have a lot of bifurcation points which may prediction of the behavior very difficult.

g) **Model of finite differences.**

In many situations it is necessary to consider the values of the variables at regular intervals of time. In economic systems the production, payments, spending and other variables are accounted yearly or monthly, some biological species have offspring at regular periods corresponding to seasons, in some educational systems the promotions of students are made by year or semester. For the models of such systems it may be convenient to use models in which the values of the variables at the end of a period depend of the values at the end of previous periods. For instance:

$x_{i,t+1} = f_i(x_{1,t}, x_{2,t}, ..., x_{n,t}, e_{1,t}, e_{2,t}, ..., e_{k,t}) \quad i = 1,2,...,n$ for the state variables and:

$y_{j,t+1} = f_i(x_{1,t}, x_{2,t}, ..., x_{n,t}) \quad j = 1,2,...,m$ for auxiliary and output variables.

The e_{i+1} are exogenous variables, $x_{i,t}$ are the state variables and $y_{i,t}$ the auxiliary variables.
The subscripts t and $t+1$ indicate that the value of the variable is taken at the end of the period t and $t+1$ respectively.
Instead of 1 some other interval length Δt can be used.
These are called **difference equations** of the **first order**. Notice the analogy with t models of differential equations (1) (2) of 4.3.1.
As it happen in differential equations, in some models difference equations of higher order may appear. In these models, values of a variable in times that differ in more than one period may exist in the same equation. As in the case of differential equations, introducing new variables may reduce the order of the equations. Thus, the equation:

$x_{t+2} = f(x_t, x_{t+1})$

can be substituted by the system of equations of first order:

$x_{t+1} = z_t$

$z_{t+1} = f(x_t, z_t)$

In some cases the higher order difference equation can be solved directly.

Example. Macroeconomic model. (See Chiang 1985)

If Y_t is the total income accumulated during the year t in a national economic system, this income will be spent in the same year, a quantity C_t in consumption and a quantity I_t in investment:

$Y_t = C_t + I_t$

Assuming that the investment is proportional to the increase in the consumption:

$I_t = b(C_t - C_{t-1})$

If the consumers spend a fraction a of their income:

$C_t = aY_{t-1}$ where $a < 1$

Substituting in the first one with the appropriate subscripts:

$Y_t = aY_{t-1} + b(aY_{t-1} - aY_{t-2})$

$Y_t - a(1-b)Y_{t-1} + abY_{t-2} = 0$

The solution of this linear second order differential equation (see the analogy with 4.1 e) is: kr^t where k is a constant and r is determined by substitution in the equation and dividing by kr^{t-2}, the result is:

$r^2 - a(1-b)r + ab = 0$

that is called the **characteristic equation** of the difference equation. From it:

$$r = \frac{a(1-b)}{2} \pm \sqrt{\frac{a^2(1-b)^2}{4} - ab}$$

there are two solutions and the general solution is a linear combination of both:

$Y_t = Ar_1^t + Br_2^t$

from the expression of r it results $r_1 > 0$ $r_2 > 0$.

If the roots are real and different, the solution may growth or decrease exponentially. If they are equal, it can be shown that the general solution is $Y_t = Ar_1^t + Btr_2^t$ that also tends to 0 or ∞. If they are complexes the solution oscillates with amplitudes convergent to 0 or diverging, except if it is exactly: $ab = 1$.

An economic system in **which the hypothesis of the model are valid,** is, in general, divergent or convergent to 0. To maintain it stable with a positive income exogenous controls are required. This is the function of central banks that control the investment by manipulation of interest rates (changing *b*) and of governments that alter the consumption manipulating the public expenditure (changing *a*).

h) **Delays.**

In some situations an output of an element of the system may adapt gradually to an input. This may be expressed by a differential equation or, more intuitively, by a difference equation (Forrester 19##). A good example is a tank with a variable inflow $I(t)$ and an outflow $O(t)$ proportional to the content $A(t)$ (it may be physically realized by a cylindrical tank with an outlet in the base through a tube with enough friction). The system equations are:

$$\frac{dA}{dt} = I(t) - O(t)$$

$$O(t) = kA(t)$$

From these:

$$\frac{dO}{dt} = k(I(t) - O(t))$$

if $I(t) > O(t)$ then $\frac{dO}{dt} > 0$ the output growths

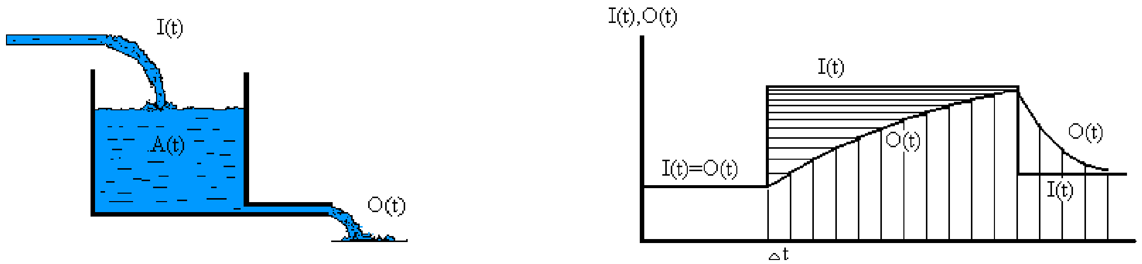
if $I(t) < O(t)$ then $\frac{dO}{dt} < 0$ the output decreases

then an equilibrium is reached when $I(t) = O(t)$ because $\frac{dO}{dt} = 0$. The output adapts itself to the input with some delay that increases as k decreases. The inverse of k, $D = 1/k$

is called delay of the system response to an input.

Using difference equations the model of the system is:

$$O(t+\Delta t) = O(t) + (I(t) - O(t))\Delta t / D$$



At each time interval the output increase is proportional to the difference between the input and output. Figure 10 .

This type of delay is called **first order delay** and is modeled by one first order differential or difference equation.

There are systems with delays of higher order. They can be modeled by a series of first order delays in which the output of one is the input of the following. The model for a third order delay is:

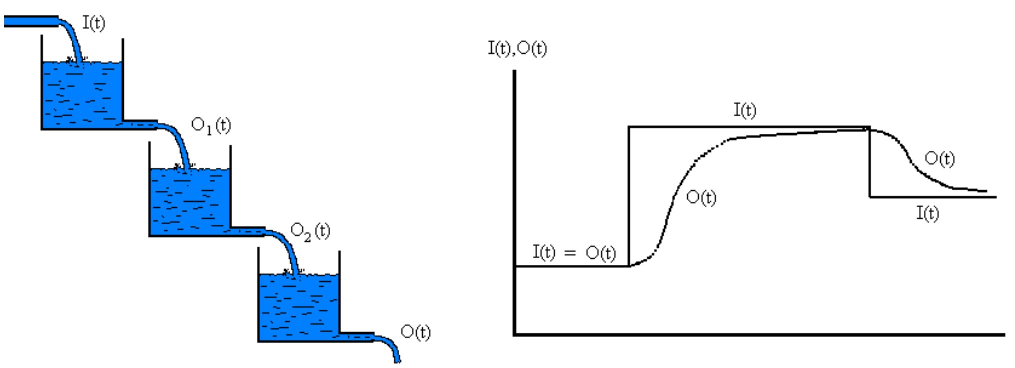


Figure 4- Third order delay

$$\frac{dO_1}{dt} = (I(t) - O_1(t))\,3/D$$

$$\frac{dO_2}{dt} = (O_1(t) - O_2(t))\,3/D$$

$$\frac{dO}{dt} = (O_2(t) - O(t))\,3/D$$

where the total delay of the system has been distributed in the three subsystems with the value $D/3$ for each one.

The corresponding difference equations are:

$$O_1(t+\Delta t) = O_1(t) + (I(t) - O_1(t))\,3\Delta t/D$$

$$O_2(t+\Delta t) = O_2(t) + (O_1(t) - O_2(t))\,3\Delta t/D$$

$$O(t+\Delta t) = O(t) + (O_2(t) - O(t))\,3\Delta t/D$$

As the order of the delay increases the outputs tends to be equal to the input displaced D in time. Figure 11.

i) **Model of self-controlled system. Feed-forward and feed-back.**

In many natural and artificial systems it is important that certain variables remain constant or have little variation, in spite of changes in other endogenous or exogenous perturbing variables. The body temperature in fowls and mammals when external temperature or internal generation of heath change, the output of a radio when the input signal varies, the speed of a turbine when the dynamo attached to it is damped by changes in the electric load, the vertical position of the human beings when they raise the arms or lift a weight up, are examples of systems in which some variables must be controlled to maintain constant the values of other variables. There are two main ways of control:

i) Through **feed-forward**, that consist in detecting the values of the **perturbing variables**, to estimate their action in the **variables to be controlled**, and to change other variables (**control variables**) that compensate the changes in the variables to be controlled. Thus, the flow in the low course of a river can be controlled by dams in its upper course and tributary rivers that are left to fill during the rainy season and to yield water during the dry season.

ii) Through **feed-back,** that consists in detecting the value of the variable to be controlled and calculate the difference between it and the desired value. This difference acts on a control subsystem that modify the action of the perturbing or control variables to compensate that difference. One of the oldest examples of an artificial regulated systems was built by Ctesibius of Aexandria about 270 bC. The objective was to maintain constant the level of water in a tank to feed a water clock that requires a constant flow of water. The entry water goes through a pipe that ends in an inverted funnel. A body floating in the tank has a conic cap that can enter into the funnel. If the level raises the floater raises and the cone enter the funnel reducing the incoming flow. If the level goes down the cone goes down and allows the flow to increase. The level is thus kept constant and the outflow in the bottom is constant. See Figure 12.

Control principles has been developed in a sophisticate mathematical theory (Barnett S, 1975).

Example. Thermal regulation of a room.

A room has gas heating that uses g m^3/sec of gas. The quantity of heat produced by hour is pg where p is a constant.. The room leaks heat at a rate proportional to the difference $T-T_e$ between the actual temperature T and the external temperature T_e. So the increase of heat in the room is $pg-k(T-T_e)$ where k is the conductivity of the boundaries of the room. The rate of temperature increase is proportional to the increase of heat accumulated in the room. The regulator subsystem decreases or increases the flow of gas proportionally to the difference $T-T_r$, where T_r is the required temperature that is to be maintained. So the model is:

$$\frac{dT}{dt}=s(gp-k(T-T_e))$$
$$g=MAX(0,r(T_r-T))$$

where k, s, p and r are constants.

k (cal/sec/^{o}C is the conductivity of the boundaries: the quantity of calories/sec that is loss for each degree of difference between the internal and external temperature.

s (^{o}C/sec/cal) is a kind of specific heat: the temperature increase of the room for each calorie produced inside.

p (cal/m3) is a heat power of the gas: the quantity of calories produced by a m^3 of gas.

$MAX(x,y)$ is a function that it is equal to its greater argument. Thus if $T>T_r$ the value of the function is 0.

r is the constant of the control system; it is equal to the quantity of gas that flows for each degree of temperature difference between the target and the room temperature.

Notice that g is an auxiliary variable that can be eliminated by substitution. The difficulty to solve the resulting equation analytically, lies in the non linear function MAX .

The simplicity get by the use of feedback can be seen in this example. To control the temperature by feed-forward implies to represent explicitly each process that may introduce changes in temperature: changes in outside temperature, opening and closing of doors and windows, entering and going out of people, turning on and off of other heating devices (lamps, irons, etc.). Besides that, the calculation of their effects on temperature and the compensatory actions by firing gas must be done. In the feedback approach the only variable to take into account is the variable to control: the temperature. This simplicity is the cause of the widespread use of feedback in physical, biological, social and economic systems.

Another important concept that can be extracted from this example is the problem of the **identification of parameters**. If, ignoring the case in which $T > T_r$, and therefore the non linear function, and g is substituted in the first equation this is reduced to the form: $T' = -hT + m$ where the h and m are functions of the parameters k, s, p (assuming that T_e, T_r and r are known). The analytical solution $T = \frac{m}{h}(1 - e^{-ht})$ corresponding to an initial temperature of 0^{o}C (any other value could be taken) may be used to estimate m and h by non linear regression. But this is useless because there are 3 unknown parameters: k, s, p in the model and only two equations that relate them with the estimated values: m and h . So, independent experiments must be performed on the system to find the structural parameters k, s, p . These problems of lack of the identification of the structural parameters by means of the observed behavior of the system appears in multi-equation models with linked variables. Some simple cases are treated in Econometric Modeling.

j) **Catastrophes**.

Many dynamic systems can be described by a smooth function $U(x_1, x_2, ..., x_n)$ of their state variables $x_1, x_2, ..., x_n$, called **potential function**, such that, the dynamic equations are expressed by :

$$x_i^{\cdot} = -\frac{\partial U}{\partial x_i} \quad i = 1,2,..,n$$

The vector with components: $(-\frac{\partial U}{\partial x_i}, -\frac{\partial U}{\partial x_i},..., -\frac{\partial U}{\partial x_i})$ is called **gradient** and points to the direction of the maximum growth of the function $U(x_1, x_2,..., x_n)$

It is clear that at a stationary point of the function U it must be: $x_i^{\cdot} = 0$, the x_i does not change, so that the system is at an equilibrium point , that must be stable, unstable or a saddle point. If $n = 2$, U represents a surface in R^3. If there is a point on this surface that move by gravity, and the friction is enough to make the moment near zero (there is no inertial effects). This point may follow a pathway whose direction is in each point opposed to the gradient, that is, it points in the direction of maximum downward slope. The projection of this point on the plane x_1, x_2 is a trajectory of the system toward the point of the lowest potential energy, that is a stable equilibrium point. The points corresponding to the maximum of U are unstable equilibrium points and the saddle points to points that are only of stable in one direction.

An important question is what happens if the function U changes a little. In mathematical terms: what happens if there is a small change in its parameters. It is possible that the nature of the minima does not change (although their positions may change a little). In this case it is said that the system is **structurally stable.** On the contrary if a minimum disappear or it is transformed into a maximum or into a saddle point the system is structurally unstable. Structural stability is important because in the real world the system undergo small random perturbations, or the measurements made on them to determine its state are subject to errors, so that only structurally stable systems correspond to entities in which is it possible to appreciate a proper identity on spite of these fluctuations.

To study structural instead of ordinary stability the space of the parameters must be considered. The potential function depends not only from the state variables but also on the vector of parameters $c = (c_1, c_2,..., c_k)$.

Example 1: If the potential function is: $U = x^3 + c_1 x$ it is clear from:

$$\frac{\partial U}{\partial x} = 3x^2 + c_1 \qquad \frac{\partial^2 U}{\partial x^2} = 6x$$

that for $c_1 < 0$ the function has a minimum at: $x = \sqrt{c_1 / 3}$ and a maximum at $x = -\sqrt{c_1 / 3}$

If c_1 is increased these two stationary points coalesce in $c_1 = 0$ that is an inflexion point. This is called a critical point, in which stability collapses. For positive values of c_1 there are not stationary points.

In general the set of values of the parameters for which the stability collapses is called **critical set**. In this case it is reduced to a point. This type of catastrophe is called fold (see Figure 1)

Example 2 : If the potential function is: $U = x^4 + c_1 x^2 + c_2 x$ then:

$$\frac{\partial U}{\partial x} = 4x^3 + 2c_1 x + c_2 \qquad \frac{\partial^2 U}{\partial x^2} = 12x^2 + 2c_1$$

The critical points of U occur when $4x^3 + 2c_1 x + c_2 = 0$ that for certain values of c_1 and c_2 has three critical points (maximum or minimum). They coalesce in two saddle points were the second derivative vanishes: $12x^2 + 2c_1 = 0$. From these two equations it results: $8c_1^3 + 27c_2^2 = 0$ that correspond to the critical set. In the c_1, c_2 plane this corresponds to a curve $c_1 = \frac{3}{2}\sqrt[3]{-c_2^2}$ which represents a cusp with the vertex in the origin. This type of catastrophe is called cusp.

There are seven types of catastrophes for systems with four or less parameters. (Andrews 1976, Poston 1978).

EXERCISES

1. Show that in an exponential growth with coefficient k the time to duplicate the quantity is $\ln 2/k$, and in an exponential decreasing process in such a time the quantity is reduced to one half.

2. The diameter of the earth is 6366 km and 29.2% is land. The actual population is approximately 5,940,000,000. The growth rate is about 1.4%. In what year will be one person by square meter?.

3. Some micro-organisms (paramecium notatum) are grown in vitro and fed at a constant rate by bacillus (bacillus piocianeis). In the tube a constant salt concentration allows the paramecium to reproduce, but nor to the bacillus. Thus, the rate of feeding is controlled and is maintained constant. The temperature is also kept fixed. The concentration of paramecia is observed daily. The data obtained are:

t days	0	1	2	3	5
Paramecia/mm^3	30	63	92	152	230
1					

If the growth were proportional to the population the growth would be exponential. Test the correction of this hypothesis by adjusting the data to a law $y = y_0 e^{kt}$ where y is the concentration of paramecia, and find y_0 and k.

(Fit the data to the linear function: $\ln y = \ln y_0 + kt$, use a statistical program or logarithmic paper).

If the experiment is pursued further the data are:

t days	5	6	7	8	9
Paramecia/mm^3	236	240	250	272	260

Assume that the growth is limited by the food: a g/cm^3. If b is the quantity of food per cm^3 need to maintain one individual and replace it to the next period, then the fraction of the food to maintain the density is by/a. The fraction available to growth is $1 - by/a$. When y grows this fraction decreases until $by_{max}/a = 1$ and $y_{max} = a/b$ is the limit density. The equation of growth is $\frac{dy}{dt} = k(1 - \frac{y}{y_{\max}})y$ that corresponds to a logistic: $y = \frac{y_{max}}{1 + ce^{-kt}}$

Use a program of non-linear fitting to find the values of: y_{max}, k, c.

Note that even a good fitting does not confirm the hypothesis of the limitation in feeding to explain the stop in growing. Suggest other hypothesis and possible experiments to take a decision.

4. Integrate the differential equation for the trajectories of example 4.3.2 d.

5. Show that the system $x' = y + x(1 - x^2 - y^2) \quad y' = -x - y(1 - x^2 - y^2)$ has a cycle limit: the circle of unit radius. (Put the equation in polar coordinates)

6. Show that a linear combination of two solutions of a linear second order differential equation is also a solution.

7. Show that in a second order linear differential equation with constant coefficients, if the characteristic equation has two equal roots r, the expression te^{rt} is a solution.

8. Compute the constants A and B for the solution of the free oscillation when the roots of the characteristic equations are equal and the oscillation starts from the distance 5 from the equilibrium point with zero velocity. It is s=4Kg./seg   m=1Kg.

9. Show that the solution of an inhomogeneous linear differential equation may be expressed by a linear combination of the solution of the homogeneous differential equation and a particular solution of the inhomogeneous.

10. An electric circuit is formed by a condenser of capacity C, a resistance R, and a solenoid of inductance L connected in series. The current is i. The electric differences of potential (d.d.p.) are:

for the condenser: $\frac{1}{C}\int idt$ (Volta´s Law)

for the resistance: iR (Ohm´s Law)

for the solenoid: $L\dfrac{di}{dt}$ (Faraday´s Law)

In a closed series circuit the sum of the d.d.p is zero. Establish the differential equation for the oscillations in this circuit.

11. Assume that between two elements of the circuit of Exercise 10, an alternate electromotive force: $E\,\mathrm{sen}\,\omega t$ is applied. Develop, for the electrical oscillations of the current i, the same analysis that was made for mechanical oscillations. Indicate the analogies between the constants and the variables of both cases: the mechanics and the electrical.

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